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THE LOCALLY CONNECTED COMPACT METRIC SPACES EMBEDDABLE IN THE PLANE

CARSTEN THOMASSEN

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We prove that a 2-connected, locally connected, compact topological space M is homeomorphic to a subset of the 2-sphere if and only if M is metrizable and contains none of the Kuratowski graphs K_5 and $K_{3,3}$.

1. Introduction

Kuratowski's theorem [2] says that a graph can be drawn (embedded) in the Euclidean plane or, equivalently, in the sphere, if and only if it does not contain any of the so-called *Kuratowski graphs* K_5 and $K_{3,3}$. A short proof can be found in [5], [6]. In this paper we generalize that result to 2-connected, locally connected, compact metric spaces. Here 2-*connected* means that the space remains connected after the deletion of any element. This connectivity condition is needed as demonstrated by a thumbtack, that is, the space obtained from the disjoint union of a disc in the plane and a straigt line segment by identifying the center in the disc with an end of the line segment. The result will be used in a forthcoming paper to prove the following extension of the classification of the 2-dimensional surfaces: First we say that a metric space is *locally* 2-*connected* if, for every element x in the space, and every neighborhood U around x, there is a neighborhood V of x contained in U such that both V and $V \setminus x$ are connected. Then we prove that a connected, locally 2-connected, compact metric space M either

Mathematics Subject Classification (2000): 05C10, 57M15 contains an infinite complete graph, or else M is homeomorphic to a surface S with holes. Moreover, M and S contain the same finite graphs.

2. Notation and basic properties of graphs and metric spaces

A *simple arc* is a homeomorphic image of the unit interval $[0,1]$, and a *simple closed curve* is a homeomorphic image of a circle. We abbreviate them SA and SCC, respectively. A *2-way infinite arc* is a homeomorphic image of a line. If A is an SA or a 2-way infinite arc, and x,y are elements of A, then $A[x,y]$ denotes the arc in A from x to y. We define $A]x,y]$ and $A]x,y[$ in the obvious way. We also use that notation when A is an SCC. Now there are two possibilities for $A[x,y]$, but it will always be clear from the context which one we think of. If A is an SA and we choose an orientation of A, then is makes sense to say that $x < y$ on A. If Q is a subset of A, then we define the supremum and infimum of Q in the obvious way.

We only consider locally connected metric spaces with more than one element. (Recall that a space is locally connected if, for every element x in the space, and every neighborhood U around x, there is a neighborhood V of x contained in U such that V is connected. Note that every open subset of a locally connected space is also locally connected.) Thus "connected" is the same as "arcwise connected". We say that a metric space M is 2-*connected* if, for any element x of M, $M \setminus \{x\}$ is connected. We also say that M is 3-*connected* if, for any elements x,y of M, $M \setminus \{x,y\}$ is either connected or has precisely two connected components, one of which is an arc from x to y. The space consisting of three SA-s joining the same two elements is also called 3-connected. These definitions of 2- and 3-connectivity extend those for graphs, see e.g. [1], [3]. A *graph* consists of a set $V(G)$ of *vertices* and a set of unordered pairs xy of vertices called *edges*. If the edge xy is present, we say that x and y are *neighbors*. The number of neighbors of x is the *degree* of x. A graph G is *bipartite* if its vertex set can be partitioned into two sets (called the *bipartite sets*) such that every edge of G joins a vertex in one of the bipartite sets with a vertex in the other. It is well-known and easy to see that a graph is bipartite if and only if it contains no odd cycle, see e.g. [1], [3]. It is also easy to see that a connected bipartite graph has only one bipartition. If G is a graph and we insert new vertices of degree 2 on some edges, we obtain a *subdivision* of G. Clearly, a graph G may be regarded as a locally connected metric space where every edge is regarded as a subspace isometric to a unit interval. If H is obtained from a subdivision of G by adding edges, then we say that H is an *extension of* G.

If M is a compact subset of the sphere, then every connected component of the complement is called a *face* of M. If M is a 2-connected graph, then every face boundary is a cycle in the graph, that is, an SCC, see e.g. [3]. The distance between two elements x, y is denoted $d(x, y)$. The set of elements of distance $< \epsilon$ from x is called the *ϵ-disc* around x and is denoted $D(x, \epsilon)$. A subset Q of M is *ϵ-dense in M* if M is the union of the ϵ-discs around the elements of Q. Consider a collection of SA-s or SCC-s. Let us say that an SA or SCC in the collection has property p. Let ϵ denote the infimum of the diameters of the $SA - s$ or $SCC - s$ with property p. Then we say that an SA or SCC with property p is *short with property* p if it has diameter less than 2ϵ. The diameter of a set Q is denoted $diam(Q)$.

Consider now a subset C of M and a connected component Q of $M \setminus C$. If M has an SA from Q to C having only the end in common with C, then that end is called a *point of attachment* of Q. The union of Q and all its points of attachments is called a *C-component of M*. If C is an SCC, then two C-components *avoid one another* if C can be divided into two $SA - s$ having only the ends in common such that one of the $SA - s$ contains all points of attachments of one of the C-components and the other SA on C contains all points of attachments of the other C-component. Otherwise, they *overlap*. If C contains four elements x, y, u, v in that cyclic order, such that x, u are points of attachments of one of the C-components, and y, v are points of attachments of the other C-component, then the C-components are *skew*. Clearly skew C-components overlap. The converse is almost true.

Lemma 2.1. *Let (M, d) be a locally connected, compact metric space M. Let C be an SCC in M, and let A, B be overlapping C-components. Then either A, B are skew or else they have the same three points of attachment.*

Proof. As M is locally connected, A has at least two points of attachment. If A has only two points of attachment, say p, q, then both arcs in C from p to q contains an element distinct from p, q which is a point of atttachment of B, and hence A, B are skew. So assume that A has at least three points of attachment, p, q, r, say. Similarly, B. has at least three points of attachment. If two of the three arcs of $C \setminus \{p, q, r\}$ contains a point of attachment of B, then clearly A, B are skew. If none of them contains a point of attachment of B, then clearly A, B have the same three points of attachment. So assume that some point of attachment of B is in $C]p, q[$, and none are in $C]p, r[$, $C]r, q[$. Also we may assume that r is is not a point of attachment of B since otherwise, A, B are skew. In other words, all points of attachment of B are in $C[p, q]$. If some two of them have a point of attachment of A strictly between them, then A, B are skew, so assume that this does not happen.

Then define a, b as the *supremum* (respectively *infimum*) of the points of attachment of B in $C[p,q]$. Then the two arcs on C from a to b show that A, B do not overlap, a contradiction which proves Lemma 2.1.

If C is an SCC in M, then we define the *overlap graph* $O(M,C)$ as the graph whose vertices are the C-components such that two vertices are neighbors if and only if they overlap. ∎

Lemma 2.2. *Let (M,d) be a locally connected, 3-connected, compact metric space M, and let C be an a SCC in M. Then $O(M,C)$ is connected.*

Proof. Let Q be a connected component of $O(M,C)$. Suppose (reductio ad absurdum) that M has a C-component, say R, which is not a vertex of Q. Let A denote the set of points of attachment of R. For every vertex of Q, all points of attachment of that vertex belong to the same component of $C \setminus A$ (plus its two ends). As Q is connected, that component of $C \setminus A$ must be the same for all vertices of Q. Let C' denote that component of $C \setminus A$ (plus its two ends). Choose an orientation of C', and let a (respectively b) be the supremum (respectively infimum) of the points of attachment on C' of the vertices of Q. As $M \setminus \{a,b\}$ is connected, M has a C-component, say R', with a point of attachment in both components of $C \setminus \{a,b\}$. We now repeat the reasoning above with R replaced by R' and obtain a contradiction which proves Lemma 2.2. ∎

3. Connectivity properties of 2-connected metric spaces

Lemma 3.1. *Let (M,d) be a locally connected, 2-connected, compact metric space, and let x be an element of M. Then M contains an SCC containing x.*

Proof. Let y be an element of M distinct from x, and let A be an SA from y to x. Let z_0 be an intermediate element on A. Let z_1' be the supremum of the elements p on $A]z_0,x]$ such that $M \setminus \{z_0\}$ has an SA from $A[y,z_0[$ to $A]z_0,x]$ hitting $A]z_0,x]$ in p. If $z_1' < x$ on A, we consider an arc A' in $M \setminus \{z_1'\}$ from $A[y,z_1'[$ to $A]z_1',x]$. Let z_1'' be the first element of A', and let A_1 be an arc from $A[y,z_0[$ to $A]z_1'',z_1']$ such that A_1 has only its first and last element in common with A. Let z_1 be the last element of A_1. We choose A_1 such that $d(z_1,z_1') < 1/2$.

Let z_2' be the supremum of the elements p on $A]z_1,x]$ such that $M \setminus \{z_1\}$ has an SA from $A[y,z_1[$ to $A]z_1,x]$ hitting $A]z_1,x]$ in p. By the definition of A', $z_2' > z_1'$ on A. This inequality is all we need A' for, and therefore we now forget about A'.

If $z_2' < x$ on A, we consider an arc A' in $M \setminus \{z_2'\}$ from $A[y, z_2'[$ to $A]z_2', x]$. Let z_2'' be the first element of A', and let A_2 be an arc from $A[y, z_1[$ to $A]z_2'', z_2']$ such that A_2 has only its first and last element in common with A. Let z_2 be the last element of A_2. We choose A_2 such that $d(z_2, z_2') < 1/4$.

We continue defining $z_3', z_3'', z_3, z_4', z_4'', z_4, \ldots$, where $z_1 \leq z_1' < z_2 \leq z_2' < \ldots$. If $z_n' < x$ on A, choose z_n in $A]z_{n-1}, z_n']$ of distance at most $1/2^n$ from z_n'. We also get arcs $A_1, A_2, \ldots$. Suppose first the sequence is infinite, that is, $z_n' < x$ on A for each $n = 1, 2, \ldots$.

By the definition of $z_1', z_2', z_3', \ldots$, the arcs $A_1, A_2, \ldots$ are pairwise disjoint (except that possibly the last element of A_n may equal the first element of A_{n+2}), and therefore $A \cup A_1 \cup A_2 \ldots$ contains a unique 2-way infinite arc containing $A_1 \cup A_2 \ldots$. We claim that this arc together with x can be chosen to be a SCC. Clearly, the sequence $z_1, z_2, \ldots$ converges to some element u, say. We must have $u = x$ since otherwise an arc in $M \setminus \{u\}$ from $A[y, u[$ to $A]u, x]$ would contradict the choice of the sequence $z_1', z_2', z_3', \ldots$. It only remains to prove that $A_1, A_2, \ldots$ can be chosen such that $diam(A_n) \to 0$ as $n \to \infty$. We postpone part of this proof as it is easier than the analogous part of the proof when the sequence $z_1', z_2', z_3', \ldots$ is finite. In this case we choose the notation such that $z_1' = x$. Now we select a sequence of arcs $A_1, A_2, \ldots$ as follows. A_1 is an arc from $A[y, z_0[$ to $A]z_0, x]$ having only it first and last element, which we denote by z_1, in common with A. Having already defined $A_1, A_2, \ldots, A_{n-1}$ with last elements $z_1, z_2, \ldots, z_{n-1}$, respectively, we define A_n with last element z_n as follows: A_n is an arc from $A[y, z_{n-1}[\cup A_1 \cup A_2 \ldots \cup A_{n-1}$ to an element z_n of $A[z_{n-1}, x]$ such that $d(z_n, x) < d(z_{n-1}, x)/2$. Furthermore, we choose A_n to be short with this property. We claim that $diam(A_n) \to 0$ as $n \to \infty$. When this claim has been proved, we complete the proof as follows: By considering an appropriate subsequence, if necessary, we may assume that the first element on A_n belongs to A or A_{n-1} but not A_{n-2}. Then $A \cup A_1 \cup A_2 \ldots$ contains a unique 2-way infinite arc which together with x form a SCC.

Suppose (reductio ad absurdum) that the claim that $diam(A_n) \to 0$ as $n \to \infty$ is false. By upscaling d if necessary, we may assume that infinitely many A_n have diameter at least 6. In each of those we pick an element u_n of distance precisely 1 from z_n. We may assume that $u_n \to u$ as $n \to \infty$. Let U be a connected open set in $D(u, 1)$ containing u. For some distinct natural numbers n, m where $n < m$, U contains an arc connecting u_n, u_m. This shows that A_m can be chosen to have diameter less than 3, contradicting the assumption that A_m is short.

In the case where $z_1', z_2', z_3', \ldots$ is infinite the same argument gives a contradiction to the fact that $A_1, A_2, \ldots$ must be pairwise disjoint. ∎

Theorem 3.2. *Let (M,d) be a locally connected, compact metric space. Then M is 2-connected if and only if, for any two elements x,y, M has a SCC through x,y.*

Proof. The "if" part is obvious. So assume that M is 2-connected, and let x,y be elements of M. Let A be an SA from y to x, and let z_0 be an intermediate element on A. We repeat the proof of Lemma 3.1. This results in an SCC C_1 containing x and an SA of the form $A[z_1,z_0]$. Similarly, there is a SCC C_2 containing y and an SA of the form $A[z_0,z_2]$. Clearly, $C_1 \cup C_2$ contains a SCC which contains both x and y. ∎

If G is a connected graph and r is a vertex or an edge of G, then r is a *cutvertex*, respectively *cutedge*, if $G - r$ is disconnected.

Corollary 3.3. *Let G be a graph with minimum degree at least 2 embedded in a locally connected, compact metric space M.*

(i) If M is connected, then G can be extended to a connected graph in M.

(ii) If M is 2-connected, then G can be extended to a 2-connected graph in M.

(iii) If M is 3-connected, then G can be extended to a 3-connected graph in M.

Proof. (i) is easy to prove by successively adding arcs between the connected components. (ii) is proved similarly: For every cutvertex x we add successively arcs between the connected components of $G - x$. This proves (ii) unless G has a cutedge $e = xy$. In that case we apply the method of Lemma 3.1 to add a finite number of arcs so that no point of e is a cutvertex. (iii) is proved by the same method. We first extend G to a 2-connected graph. Then we consider any set $\{x_1,x_2\}$ of vertices whose deletion from the current graph results in a disconnected graph, and we add arcs between the connected components. There is only a problem for pairs of edges e_1,e_2 whose deletion results in a disconnected graph. However, an easy modification of the proof of Lemma 3.1 again gives the desired result. ∎

4. Embedding flat spaces in the sphere

We say that a metric space is *flat* if it contains none of the Kuratowski graphs K_5 or $K_{3,3}$.

Lemma 4.1. *Let (M,d) be a locally connected, compact metric space M, and let C be an SCC in M. If M contains none of the Kuratowski graphs K_5 and $K_{3,3}$, then $O(M,C)$ is bipartite.*

Proof. Suppose (reductio ad absurdum) that $O(M,C)$ is nonbipartite. Then it has an odd cycle $Q_1,Q_2,\ldots,Q_{2k+1},Q_1$ where the indices are expressed modulo $2k+1$. That is, Q_i and Q_{i+1} overlap for $i=1,2,\ldots,2k+1$. Using Lemma 2.1 we can find a connected subset Q'_i of Q_i such that Q'_i and Q'_{i+1} overlap for $i=1,2,\ldots,2k+1$, and such that $C\cup Q'_1\cup Q'_2\cup\ldots\cup Q'_{2k+1}$ is a graph. That graph is nonplanar and contains therefore, by Kuratowski's theorem, a subdivision of one of K_5 and $K_{3,3}$, a contradiction which proves Lemma 4.1. ∎

If M and C are as in Lemma 4.1, and x,y are in $M\setminus C$, then C *separates* x and y if x and y belong to distinct components of $M\setminus C$. If, in addition, M is 3-connected, then $O(M,C)$ is connected by Lemma 2.2, and so $O(M,C)$ has only one bipartition. Now if the components containing x and y belong to distinct parts of $O(M,C)$, then C *strongly separates* x and y. So a flat 3-connected space satisfies a kind of Jordan Curve Theorem. This can be refined in terms of graphs: Consider a 3-connected graph G in a flat 3-connected, locally connected, compact metric space M. Let C be any facial cycle of G. As G is 3-connected, G has only one C-component. The union of C and all C-components in M which C strongly separates from $G\setminus C$ is called the *face* of G corresponding to C. It is also called *the interior of* C and is denoted $Int(C)$. The union of C and all other C-components is called *the exterior of* C and is denoted $Ext(C)$. These separation properties play a crucial role in the main result below. First we express strong separation in terms of graphs.

Lemma 4.2. *Let (M,d) be a flat 3-connected, locally connected, compact metric space, and let x,y be elements of M. Then some SCC in M strongly separates x,y if and only if M contains a 3-connected planar graph G containing x,y such that G has no facial cycle containing x and y.*

Proof. Suppose first that M contains a 3-connected planar graph G containing x,y such that G has no facial cycle containing x and y. A result of Whitney says that a 3-connected planar graph has only one embedding in the sphere, see e.g. [3], or [4], or [5]. Then the union of G and a new edge from x to y is nonplanar, by Whitney's theorem. It contains therefore a subdivision of one of the Kuratowski graphs K_5 and $K_{3,3}$, by Kuratowski's theorem. If we remove the edge xy from this Kuratowski graphs we obtain a graph G' in M such that G' contains a cycle which strongly separates x and y in G', and therefore also in M.

Suppose next that M has an SCC C which strongly separates x and y. Let $Q_1,Q_2,\ldots,Q_m$ be C-components such that Q_1 contains x, Q_m contains y, and Q_i,Q_{i+1} overlap for $i=1,2,\ldots,m-1$. Using Lemma 2.1 and Lemma 3.1

we can find a 2-connected graph G' containing C, x, y such that x and y belong to distinct parts of $O(G', C)$. By Corollary 3.3, we can extend G' to a 3-connected graph G, and clearly x and y belong to distinct parts of $O(G, C)$ In particular, they cannot be on the same facial cycle of G. This proves Lemma 4.2. ∎

Theorem 4.3. *Let M be a locally connected, 2-connected, compact topological space M. Then M is embeddable in the 2-sphere if and only if M is metrizable, and contains none of the Kuratowski graphs K_5 and $K_{3,3}$.*

Proof. The "only if" part is trivial. So assume that (M, d) is a locally connected, 2-connected, compact metric space which contains none of the Kuratowski graphs K_5 and $K_{3,3}$. Clearly, M contains an SCC which we denote by C_0. We draw C_0 as the equator on a sphere and denote the drawing by C'_0. By Lemma 4.1, C_0 divides M into two parts M_0, M_1 having precisely C_0 in common, where each part corresponds to a bipartite class of the overlap graph $O(M, C_0)$. We shall now embed one of the parts on the southern hemisphere and the other part on the northern hemisphere. It suffices to concentrate on the northern hemisphere which we now think of as a square, which we also denote by C'_0, and its interior in the Euclidean plane. We shall extend C'_0 to an embedding of M_0 inside C'_0. If $M_0 = C_0$, there is nothing to prove. So assume that C_0 can be extended to three arcs in M_0 connecting the same two elements x, y. We denote the union of these three arcs by G_1. We extend C'_0 to a drawing G'_1 of G_1 inside C'_0. All edges inside C'_0 are polygonal. We are going to define a homeomorphism of M_0 to $Int(C'_0)$. We let the vertices of G_1 be mapped to the vertices of G'_1. Also, the edges of G_1 are going to be mapped to the edges of G'_1, but so far, we do not decide precisely how. Moreover, the faces of G_1 are going to be mapped to the faces of G'_1, but so far, we do not decide precisely how.

We consider first the case where M_0 is *almost* 3-*connected*, that is, if $\{x, y\}$ separates M_0 nontrivially, then both of x, y are on C_0. In other words, the union of C_0 and any C_0-component in M_0 is 3-connected. If R is an SCC in M_0 such that $R \cup C_0$ is a graph, then we define $int(R)$ as follows: By Corollary 3.3, $R \cup C_0$ can be extended to a 3-connected graph H in M. We may assume that in the planar drawing of H, R is in the interior of C_0. Then C_0 is in the exterior of R. Hence, all paths of $C_0 \setminus R$ are in the same bipartite class of $O(H, R)$ and hence also in the same bipartite class of $O(M, R)$. The subset of M in the other bipartite class of $O(M, R)$ is denoted $int(R)$. We also put $Int(R) = int(R) \cup R$. A graph H in M_0 containing C_0 is called *almost* 3-*connected* if every subgraph of H consisting of C_0 and a C_0-component in H is 3-connected. In particular, H is 2-connected and every separating set of two vertices (if any) is a subset of C_0. H can be drawn in the plane such that C_0 is the outer cycle, and it is an easy consequence of Whitney's theorem that any such drawing (embedding) is unique up to homeomorphism of the plane.

We define a sequence of finite almost 3-connected graphs $G_1 \subset G_2 \ldots$ in M_0 and a corresponding sequence of drawings $G'_1 \subset G'_2 \ldots$ in $Int(C'_0)$. (Recall that all edges in $Int(C'_0)$ are polygonal arcs.) When G_n and G'_n have been defined we let f map G_n onto G'_n. But again, f is only defined on $V(G_n)$. An edge in $E(G_n)$ will be mapped onto the corresponding edge in $V(G'_n)$, but we allow ourselves the freedom to decide later in the proof precisely how the mapping is defined. Similarly for the faces.

We describe informally the idea of the proof. We define f on $V(G_1) \cup V(G_2) \ldots$. We choose the sequence $G_1, G_2, \ldots$ such that $V(G_1) \cup V(G_2) \ldots$ is dense in M_0 so that there is at most one way of extending f to M_0. The image M'_0 of M_0 has faces (that is, connected components in the complement). These faces have boundaries which we need to find in M_0. For, if some face boundary is disjoint from $G_1 \cup G_2 \ldots$, then we encounter the difficulty that $G'_1 \cup G'_2 \ldots$ may "rotate" around the face, and we cannot define f on that face boundary. Even if if some face boundary has some elements in common with $G_1 \cup G_2 \ldots$, then there is the risk that $G'_1 \cup G'_2 \ldots$ may "oscilate" close to the face. Therefore, we shall choose the sequence $G_1, G_2, \ldots$ such that $G_1 \cup G_2 \ldots$ contains a dense subset of each face boundary. If two face boundaries intersect, then the sequence $G_1, G_2, \ldots$ may indicate that the two faces are just one face. Therefore, we choose the sequence $G_1, G_2, \ldots$ such that $G_1 \cup G_2 \ldots$ separates any two distinct face boundaries. More precisely, any two face boundaries will be separated by a cycle in one of the graphs $G_1, G_2, \ldots$.

We also need to be careful with the drawing inside C'_0 even though all edges are polygonal. We need to reserve space for the faces of M'_0 so that one face is not by mistake divided into two. We also need to make sure that the drawing does not get faces that should not be there. In other words, if a face in G'_n has small diameter, then the same should hold for the corresponding face in G_n. When these conditions are satisfied, then it is easy to prove that f has a unique extension to a homeomorphism of M_0 onto a subset of $Int(C'_0)$.

We now argue formally. Suppose that we have defined an almost 3-connected graph G_{n-1} in M_0 and a drawing G'_{n-1} of G_{n-1} such that all edges are polygonal. We also assume that we have defined a finite sequence A_{n-1} such that A_{n-1} is $1/(n-1)$-dense in M_0 and contains a $1/(n-1)$-dense subset of G_{n-1}. Repetition of elements are allowed in A_{n-1}. Let A^2_{n-1} be a finite sequence of unordered pairs of elements of A_{n-1} such that each pair appears at least once. We consider the first pair x,y in A^2_{n-1}. We use the elements x,y to extend G_{n-1},G'_{n-1} into G_n,G'_n.

We may assume that x is in G_{n-1}, since otherwise, we first use Theorem 3.2 to add an SA which contains x and has precisely its ends in common with G_{n-1}, and then we use Corollary 3.3 to extend the union of that arc and G_{n-1} into an almost 3-connected graph. Similarly for y. We consider a short arc P in M_0 joining x,y. Then $P\setminus G_{n-1}$ consists of at most countably many arcs. Let R be one of these arcs. We add it to G_{n-1} if it satisfies the following: The ends of R divide some facial cycle of G_{n-1} into two paths, say P'' and P'''. If both $Int(R\cup P'')$ and $Int(R\cup P''')$ have diameter at least $1/n$, then we add R to G_{n-1}. In this way we add only finitely many arcs to G_{n-1}, and the resulting graph is also denoted G_{n-1}.

Consider also the case where M has an SCC C which strongly separates x,y. Let H_x be the union of C and the elements that C does not strongly separate from x. Similarly for H_y. Now choose C such that $\min\{d(x,H_y),d(y,H_x)\}$ is large (that is, at least half the supremum). Then $C\setminus G_{n-1}$ consists of at most countably many arcs. Let R be one of these arcs. We add it to G_{n-1} if it is in M_0 and satisfies the following: The ends of R divide some facial cycle (distinct from C_0) of G_{n-1} into two paths, say P'' and P'''. If both $Int(R\cup P'')$ and $Int(R\cup P''')$ have diameter at least $1/n$, then we add P' to G_{n-1}. In this way we add only finitely many arcs to G_{n-1}, and the resulting graph is also denoted G_{n-1}.

The resulting graph is then extended to an almost 3-connected graph by Corollary 3.3, and this is denoted G_n. We extend A_{n-1} to a finite sequence which is both $1/n$-dense in M_0 and in G_n. Also, in A^2_{n-1} we move the pair x,y to the end and extend it to a sequence A^2_n of unordered pairs of elements of A_n such that each pair appears at least once. We have now defined the sequence $G_1,G_2,\ldots$.

We now claim

(1) Two elements x,y in M_0 and not both on C_0 are strongly separated by an SCC in M if and only if there exists a natural number n such that x,y are strongly separated by some cycle of G_n.

Proof of (1): The "if" part is trivial so we prove the "only if" part. Let G satisfy the conclusion of Lemma 4.2, and let n be such that G_n has at least two elements in common with G. We may assume that x is not in C_0. Then G_n has a unique cycle C_1 such that $int(C_1)$ contains x and is minimal with this property. We may assume that y is in $Int(C_1)$. Let C be a facial cycle of G_n such that x and y are in $Int(C)$. We also consider G'_n which is drawn in $Int(C'_0)$.

We claim that G contains an arc P such that P is in $Int(C)\setminus\{x,y\}$ and such that one of the two cycles R_1,R_2 in $C\cup P$ containing P has x in the exterior, and the other has y in the exterior. To prove this we shall extend G'_n to a drawing of $G_n\cup G$ by drawing each G_n-component of $G_n\cup G$ separately. There may be countably many components, but if we enumerate them, then the sequence of diameters tends to 0 as n tends to infinity. Also, they can be drawn in $Int(C'_0)$ such that the sequence of diameters tends to 0 as n tends to infinity. This proves the claim that the above arc P exists since otherwise we could add the arc xy to $G_n\cup G$ and draw the nonplanar graph $G\cup\{xy\}$ in the plane, a contradiction.

Let $\epsilon=\min\{d(x,Int(R_1)),d(y,Int(R_2))\}$. Choose m such that $1/m<\epsilon/10$. We may assume that G_m has a facial cycle $R\subset Int(C)$ such that x,y are in $Int(R)$. We now repeat the argument of the previous paragraph with G_m instead of G_n and P instead of G to conclude that P contains an arc S such that S is in $Int(R)\setminus\{x,y\}$ and such that one of the two cycles in $R\cup S$ containing S has x in the exterior, and the other has y in the exterior. (Otherwise, we can extend G'_m to a drawing of $G_m\cup P$ in such a way that we can also add the edge xy and thereby obtain a drawing in the plane of the nonplanar graph $G_n\cup P\cup\{xy\}$.) Now let x_1,y_1 be in A_m such that $d(x,x_1)<1/m$ and $d(y,y_1)<1/m$. We may choose x_1,y_1 such that they are in $Int(R)$. By choosing a larger m, if necessary, we may assume that x_1,y_1 is the first pair in A_m^2. By the way in which G_{m+1} is constructed, it contains an arc R_1 in $Int(R)$ which shows that x,y are not on or inside a common facial cycle of G_{m+1}. Then G_{m+1} contains a cycle which strongly separates x and y. This proves (1).

Consider now two elements x,y in M_0 such that, for each natural number n, G_n has a facial cycle R_n distinct from C_0 such that $x,y\in Int(R_n)$. Clearly, $Int(R_1)\supseteq Int(R_2)\supseteq\ldots$, and hence $Int(R_1)\cap Int(R_2)\cap\ldots$ is a compact, connected set called a *face boundary containing* x,y. Note that there may be two but not three face boundaries containing x,y as each G_n is almost 3-connected. If there are two face boundaries containing x,y, then $\{x,y\}$ separates M_0, and one of the components is an arc which is part of an edge in each G_n for n sufficiently large. It follows from (3) below that x,y are contained in precisely two face boundaries where each of these has an element not contained in the other. But, at present we do not exclude the possibility that some face boundary may be a proper subset of another face boundary. But, if that happens, then the small face boundary is an arc of some edge of some G_n. Hence there are only countably many face boundaries that are proper subsets of some other face boundary. We shall now prove that there are also only countably many maximal face boundaries.

We say that C_0 is a maximal face boundary. By (1), any other maximal face boundary may be described as a maximal subset of M_0 which is not contained in C_0 and which has the property that no two elements of the set are strongly separated by a SCC in M. Therefore, the maximal face boundaries are independent of the particular way in which we define G_n, A_n. We define the *reduced diameter* of the face boundary as $limsup\{diam(R_n)\}$. (At this stage of the proof we cannot exclude the possiblility that this definition of reduced diameter may depend on the way in which we define G_n, A_n, though.)

(2) For each positive ϵ, M_0 has only finitely many maximal face boundaries of reduced diameter $> \epsilon$.

Proof of (2): If the above-mentioned face boundary has reduced diameter $> \epsilon$, then $diam(R_n) > \epsilon$ for infinitely many n. For each such n let x_n, y_n, z_n be elements of R_n such that the distance between any two of them is at least $\epsilon/4$. By taking a subsequence, if necessary, we may assume that x_n tends to x, y_n tends to y, and z_n tends to z, as n tends to infinity. We call x, y, z an ϵ-*triple* for the face boundary under consideration. Now suppose (reductio ad absurdum) that there are infinitely many face boundaries of reduced diameter $> \epsilon$. We select an ϵ-triple in each of them. Then there is an infinite sequence of distinct face boundaries of reduced diameter $> \epsilon$ and three elements x_0, y_0, z_0 of M_0 such that the first (respectively second, respectively third) element of the ϵ-triple of the face boundaries converges to x_0 (respectively y_0, respectively z_0). Consider now in M three pairwise disjoint, open, connected neighborhoods U_x, U_y, U_z containing x_0 (respectively y_0, respectively z_0). Then some G_n has three distinct facial cycles R_n, S_n, T_n each of which intersects each of U_x, U_y, U_z. The intersection $G_n \cap U_x$ may have countably many components. But, their diameters tend to 0. Therefore it is possible to add to G_n a finite number of arcs in U_x (which we think of as new edges) such that some component in U_x of the enlarged graph intersects each of R_n, S_n, T_n. We also add these arcs as edges to G'_n. We do the same for U_y, U_z. Then we pick a point in each of the faces bounded by R'_n, S'_n, T'_n and it is now easy to extend these three points to a planar drawing of $K_{3,3}$, a contradiction which proves (2).

By (2), there are only countably many maximal face boundaries. Let $F_1, F_2, \ldots$ be an enumeration of the maximal face boundaries.

We may assume that, for each natural number n, A_n contains a subset which is $1/n$-dense in F_n. (Note, that if we enlarge A_n, then the maximal face boundaries do not change, as we noted after (1). The reduced diameters might change, but they were only used to enumerate the face boundaries.)

Consider now any face boundary F. (F is not necessarily maximal.) Let n be so large that G_n contains at least four elements of F. (Note that if F is not maximal, then F is an SA contained in each G_q for q sufficiently large.) As no cycle of G_n separates any two elements of F, it follows that G_n has a facial cycle containing all of $G_n \cap F$. This cycle induces a cyclic ordering of $G_n \cap F$. For any four elements of $G_n \cap F$, there is precisely one way to add two edges between them so that we obtain a nonplanar graph. From this it follows that the cyclic ordering of $G_{n+1} \cap F$ in G_{n+1} induces the cyclic ordering of $G_n \cap F$ in G_n. So we obtain a cyclic ordering of $(G_1 \cup G_2 \cup \ldots) \cap F$. We call this the F*-ordering*. We shall now prove that this F-ordering satisfies the following density condition.

(3) Let F be a face boundary which is not necessarily maximal, and let $x, y \in (G_1 \cup G_2 \cup \ldots) \cap F$. Then F contains two other elements u, v such that u, x, v, y, u occur in that cyclic ordering in the F-ordering.

Proof of (3): Let n be so large that G_n contains x, y. Let R_n be the face boundary in G_n corresponding to F. That is $F = Int(R_n) \cap Int(R_{n+1}) \cap \ldots$. For each $m \geq n$, let P_m, Q_m be the two paths in R_m joining x, y. We may assume that $Q_m \cap F = \{x, y\}$ for each $m \geq n$. Choose v_m in $A_m \cap Q_m$ such that $d(v_m, x) > d(x, y)/3$ and $d(v_m, y) > d(x, y)/3$. By taking a subsequence, if necessary, we may assume that $v_m \to v$ as $m \to \infty$. As $Q_m \subset Int(R_m) \subset Int(R_n)$ for each $m \geq n$, it follows that $v \in Int(R_n)$. Hence $v \in F$. As $A_m \cap F$ is $1/m$-dense in F, and $A_m \subset G_m \cup G_{m+1}, \ldots$, we may choose u_m in $A_m \cap F \cap R_m$ and hence in $A_m \cap F \cap P_m$ such that $u_m \to v$ as $m \to \infty$. Put $\delta = d(v, Q_n)$. As v is in F, and $Q_n \cap F = \{x, y\}$, it follows that $\delta > 0$. Now let U be a connected open set in M which contains v and has diameter $< \delta/4$. For some $m > n$, U contains u_m, v_m. Assume without loss of generality that u_m, v_m is the first pair in A_m^2. As M has an arc of diameter $< \delta/4$ connecting u_m, v_m, we add to G_m an arc of diameter $< \delta/2$ connecting P_m, Q_m. As this arc does not intersect Q_n, by the definition of δ, it follows that x', y' are not on the same face boundary in G'_{m+1}. This contradiction proves (3).

As noted previously, any face boundary F which is not maximal is an SA in G_n when n is sufficiently large. If we apply (3) to F, where x, y are the ends, then we obtain a contradiction. Hence every face boundary is maximal. In other words, no face boundary is a subset of another face boundary.

(4) Let F be a face boundary, and let $x, y \in F$. Then the metric space which is the union of M and an SA R from x to y (and otherwise disjoint from M) contains none of the the Kuratowski graphs K_5 and $K_{3,3}$.

Proof of (4): Suppose (reductio ad absurdum) that $F \cup R$ contains a graph H_1 isomorphic to one of K_5 and $K_{3,3}$. Then H_1 contains R. Let x_1, y_1 be the ends of the edge in H_1 containing R. Let H_2 be H_1 minus the edge x_1y_1. Then H_2 has a cycle which strongly separates x_1 and y_1 and hence also x and y.

This contradiction to (1) proves (4).

(5) Let F be a face boundary, and let x_1, x_2, x_3, x_4 occur in that cyclic order on F. Then M has no two disjoint SA-s R_1, R_2 such that R_i joins x_i and x_{i+2} for $i = 1, 2$.

Proof of (5): Suppose (reductio ad absurdum) that R_1, R_2 exist. We apply repeatedly (4). First we add to M an SA from x_1 to x_2 which is otherwise disjoint from M. It is easy to see that the resulting space has a face boundary containing x_1, x_2, x_3, x_4 in that cyclic order. Then we also add SA-s from x_2 to x_3, from x_3 to x_4, from x_4 to x_1, and from from x_1 to x_3. Then we choose a point x_5 on the edge x_1x_3 and add also the edges x_5x_2, x_5x_4. Now the new edges together with R_1, R_2 form a K_5, a contradiction to (4).

We have described the sequence $G_1, G_2, \ldots$. In particular, we have discussed how G_{n-1} is extended to G_n. However, we shall now modify and refine the definition of G_n. We focus on the drawings of $G'_1, G'_2, \ldots$. The geometric details are easy since all edges are polygonal. However, we wish to reserve space for the face boundaries of M_0 so that they become discs, and we want all other facial cycles to have small diameter. So before we denote the current graphs by G_n, G'_n we further extend the current G_{n-1}, G'_{n-1} as follows. (That is, we still use the procedure for obtaining the graphs G_n, G'_n but not until we have further extended G_{n-1}, G'_{n-1} as described below.)

Consider first a facial cycle R, R' in the current G_{n-1}, G'_{n-1} and a face boundary F inside it whose reduced diameter is $> 1/n$ such that F has at least two elements in common with R. We extend these two elements of $F \cap R$ to a set $S(R, F, n)$ as follows: If $F \cap R$ is finite, then $S(R, F, n) = F \cap R$. If $F \cap R$ is infinite, then $S(R, F, n)$ is a finite subset of $F \cap R$ such that any other element of $F \cap R$ has distance $< 1/n$ to one of $S(R, F, n)$. Any arc of $R \cap F$ which joins two elements of $S(R, F, n)$ is also contained in $S(R, F, n)$. Then we add a finite number of simple arcs inside R' which, together with $S(R, F, n)'$ form an SCC. We may think of the new arcs as edges. We call these *pseudoedges* in G'_{n-1} as they do not correspond to edges in G_{n-1}. No point inside the SCC consisting of the pseudoedges and some $SA's$ in R' will ever be used for some G'_m. If, at some later stage we add to the current G_m an element z of A_m which is also on F, then we put it on a pseudoedge. If we later add an element z on the current G_m to $S(R, F, m)$, then we enlarge the space reserved for the face boundary by replacing part of a pseudoedge by an appropriate arc containing z'. In this way we take care of all the face boundaries in M_0.

We now try to make all faces in G'_n (minus the space reserved for the face boundaries in G_n) small. For this we consider a large collection of vertical lines and horizontal lines inside C'_0 which together bound small squares. (By making small displacements we avoid new vertices having degree 4.) G'_{n-1} divides these lines into segments. If we could add all these segments as edges (or pseudoedges) to G_{n-1}, G'_{n-1}, then the faces in G'_{n-1} would be small. So let us consider one of the segments which we are unable to add. Choose x, y in G_{n-1} such that the segment which we want to add joins x', y' in G'_{n-1}. (Possibly one or both of x', y' are on pseudoedges. But then we use (3) to find x or y (or both) and we use Corollary 3.3 to make sure that x, y are in G_{n-1}.) Let R, R' denote the corresponding facial cycle. Let R_1, R_2 be the two segments of R from x to y. The reason that we cannot add an arc from x to y is that there is at least one face boundary of M_0 which intersects both $R_1 \setminus \{x, y\}$ and $R_2 \setminus \{x, y\}$. (To see this, assume first that some R-component Q, say, in $Int(R)$ has a point of attachment on each of $R_1 \setminus \{x, y\}$ and $R_2 \setminus \{x, y\}$. Let y_i (respectively x_i) be the supremum (respectively infimum) of the points of attachment of Q on R_i for $i = 1, 2$. Then some face boundary contains y_1 and y_2 unless one (or both) of these equals y. Similarly, one of x_1, x_2 equals x. But then the above mentioned arc can be added. We argue similarly if each R-component in $Int(R)$ has all point of attachment on just one of $R_1 \setminus \{x, y\}$ and $R_2 \setminus \{x, y\}$.) If there are only finitely many such face boundaries, then we add the corresponding pseudoedges. If, however, there are infinitely many, then we add them one by one, and we stop when any new such face boundary has to be added inside a face of the current graph G'_{n-1} (plus the pseudoedges) of diameter $< 1/n$ close to x' or y'. By the proof of (2) this will happen in a finite number of steps. We also add all face boundaries between any two of those that we have already added (that is, those which are not close to x', y'). Having done that we can add straight line segments between consecutive face boundaries. As we do not want to deal with graphs of connectivity less than 2, we first select the ends of these segments to be added. They exist by (3). Then we use Corollary 3.3 to include these in the current graph. Then we add the line segments. In this way we make sure that all face boundaries of G'_n except those corresponding to face boundaries of M_0 are small.

We now extend f to a map from M_0 to $Int(C'_0)$. As f is defined on the vertices of each G_n, f is also defined on $A_1 \cup A_2 \ldots$ and hence it has a unique extension to $G_1 \cup G_2 \cup \ldots$. Consider now an element x not in $G_1 \cup G_2 \ldots$.

Then x is inside a facial cycle R_n of G_n and is going to be mapped into some point $x' = f(x)$ of $Int(R'_n)$. If $Int(R'_n)$ has diameter $> 1/n$, then there are pseudoedges in $Int(R'_n)$, and these pseudoedges divide $Int(R'_n)$ into faces reserved for faces of M' and other faces of diameter $< 1/n$. As x' is going to be in one of these faces, there is a unique choice for x'.

We have now defined f and it remains to be proved that f is $1-1$ and continuous. To prove that f is $1-1$, let x,y be distinct elements of M_0. If x,y are separated by some cycle of some G_n or if x,y both belong to some G_n, then it follows from the definition of f, that $f(x) \neq f(y)$. So we may assume that M_0 has precisely one face boundary F, say, which contains x and y. In other words, for each n, there is precisely one facial cycle R_n of G_n such that x,y are in $Int(R_n)$. Let x_n, y_n be in $F \cap (A_1 \cup A_2 \cup \ldots)$ such that x_n tends towards x, and y_n tends to y as n tends towards infinity. For some $m > n$, x_n is in G_m, that is, x_n is in R_m. For notational convenience, assume that x_n is in R_n. Similarly, we assume that y_n is in R_n. As M is locally connected, M contains disjoint connected neighborhoods U_x, U_y of x,y, respectively, each of diameter at most $d(x,y)/100$. Put $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$. By taking a subsequence, if necessary, we may assume that $X \subset U_x$ and $Y \subset U_y$. By (5), R_n contains two disjoint minimal SA's $R_n(x), R_n(y)$ such that $X \cap R_n \subset R_n(x)$, and $Y \cap R_n \subset R_n(y)$.

We choose u_n, v_n in $R_n \setminus (R_n(x) \cup R_n(y))$ such that u_n, x_n, v_n, y_n occur in that cyclic order on R_n and such that the distance from any of u_n, v_n to any of x,y is at least $d(x,y)/4$. By taking a subsequence, if necessary, we may assume that u_n tends to u, and v_n tends to v as n tends to infinity. Clearly $u, v \in F$. As $F \cap (A_1 \cup A_2 \cup \ldots)$ is dense in F, it contains a sequence $u''_1, u''_2, \ldots$ converging to u, and a sequence $v''_1, v''_2, \ldots$ converging to v. We choose the notation such that $u''_n, v''_n \in R_n$ for each $n = 1, 2, \ldots$. As X, Y are subsets of F, every R_n-component in $Int(R_n)$ is attached to an arc of $R_n \setminus (X \cup Y)$. The union of the arc $R_n \setminus (X \cup Y)$ (including its two ends) containing u_n and the R_n-components in $Int(R_n)$ attached to that arc is denoted $C(u,n)$. We define $C(v,n)$ analogously. As $C(u,n)$ is compact and contains the vertices $u_n, u_{n+1}, \ldots$, it follows that $u \in C(u,n)$. Similarly $v \in C(v,n)$.

As $u''_1, u''_2, \ldots$ tends to u, and $v''_1, v''_2, \ldots$ tends to v, there is an $m > n$ such that $u''_m \in C(u,n)$ and $v''_m \in C(v,n)$. Hence u''_m, x_m, v''_m, y_m occur in that cyclic order on R_m.

By repeating the above reasoning with u''_m, v''_m instead of x_n, y_n, and x_q, y_q instead of u_m, v_m we conclude that x,y belong to distinct R_m-components and hence $f(x) \neq f(y)$. Hence f is $1-1$.

To prove that f is continuous, we consider any element x in M_0 and any natural number n, and we show that there exists a natural number m

such that all elements in M_0 of distance $< 1/m$ from x are mapped into elements of distance $< 1/n$ from x'. In G'_n plus the pseudoedges, all faces, except those that are reserved for face boundaries of M', have diameter $< 1/n$. Either precisely one of these faces contains x' and is contained in the disc around x' of radius $1/n$ or else x' is in G'_n and is on the boundary of two or more faces of G'_n. As the former case is easier than the latter, we consider only the latter. Let $R_1, R_2, \ldots, R_q$ be the facial cycles of G_n containing x, and let m be so large that the disc around x of radius $1/m$ is contained in $Int(R_1) \cup Int(R_2) \ldots \cup Int(R_q)$. Then this disc is mapped into the corresponding disc around x' unless some faces containing x', for example $Int(R'_1)$, has diameter $> 1/n$. But then there are pseudoedges in $Int(R'_1)$, and they divide $Int(R'_1)$ into faces such that at most two of these faces have x' on the boundary and contain points of M'_0. For those faces in $Int(R'_1)$ which contain points of M'_0 but do not have x' on the boundary, we consider the union of R_1 and all those R_1-components that are mapped into the faces which contain points of M'_0 but do not have x' on the boundary. They form a compact set to which x' has a positive distance δ. If we choose m so large that $1/m < \delta$, we are done. This completes the proof when M_0 is almost 3-connected.

We shall now consider the case where M_0 is 2-connected but not almost 3-connected. We consider again C_0 and M_0. If some C_0-component has presisely two points of attachment, then we call those two points of attachment, a *maximal 2-separator*. If $S_1 = \{x_1, y_1\}$ is another separating set of two elements, then $M_0 \setminus S_1$ has precisely one component which contains $C_0 \setminus S_1$. If $S_2 = \{x_2, y_2\}$ is a separating set not intersecting that component, then we say that S_1 *is greater than* S_2. Thus the separating sets with two elements are partially ordered, and any set which is maximal in this ordering is also called a *maximal 2-separator* It is easy to see that the separating sets with two elements are inductively ordered. By Zorn's lemma, every separating set with two elements is smaller than some maximal 2-separator. For each maximal 2-separator $S = \{x, y\}$, we let R_S be an arc joining x, y in some S-component of M_0 disjoint from C_0 and we delete from M_0 all S-component of M_0 disjoint from C_0, except that we keep R_S. The resulting subspace of M_0, which we call M_1, is almost 3-connected, and we apply the first part of the proof to that subspace with a small modification: When we draw R'_S, we also draw two artificial edges close to R'_S such that R'_S (and nothing else) is in the interior of these two artificial edges. We make sure that the diameter of the union of these two artificial edges is less than 1. When we have completed the drawing of M'_1, we consider any two artificial edges forming a simple closed curve in $Int(C'_0)$. We also add the two artificial edges to the union of S and all S-component of M_0 disjoint from C_0, and we let either of these two artificial edges together with R_S play the role of C_0. Then we repeat the above argument at most countably many times. In the n'th iteration we make sure that the distance in $Int(C_0')$ between the two points in any 2-separator is $<1/n$ and that the diameter of the two corresponding artificial edges is also $<1/n$.

We shall argue why all of M_0 is drawn inside C_0' in this way. Consider again a 2-separator $S=\{x,y\}$ and a 2-separator $R=\{u,v\}$ smaller that S. By Theorem 3.2, M_0 has two disjoint SA-s Q_1,Q_2 connecting R and S. If $M_0\setminus\{x,y,u,v\}$ has no arc between Q_1 and Q_2, then we say that R,S are *similar* 2-*separators*. If $M_0\setminus\{x,y,u,v\}$ has an arc Q_3 between Q_1 and Q_2, then $Q_1\cup Q_2\cup Q_3$ has an element w such that $d(w,u)=d(w,v)$. We call this a *center between* R,S. We claim that, between any two 2-separators, there are only finitely many pairwise nonsimilar 2-separators. For otherwise, there is an infinite increasing or decreasing sequence of nonsimilar 2-separators between some two fixed 2-separators. As M is 2-connected, there is a positive ϵ such that the distance between the two elements in each 2-separator is $>\epsilon$ Between any two consecutive of these we pick a center. A subsequence of these centers converges towards an element z, say. Now a connected neighborhood in $D(z,\epsilon/3)$ shows that M has an SA joining two centers but missing each of the 2-separators, a contradiction. This proves the claim that, between any two 2-separators, there are only finitely many pairwise nonsimilar 2-separators. Therefore, we draw every 2-separator in the process above. If some element, say x, in M_0 is not drawn, then there is an infinite decreasing sequence of pairwise nonsimilar 2-separators separating x from C_0. But then we obtain a contradiction as above unless the elements in these 2-separators converge towards x. We then define x' as the limit of the corresponding 2-separators inside C_0', and hence x is drawn after all. This completes the proof. ∎

In [4] it was proved that a compact, 3-connected, locally connected subset of the sphere has only one embedding in the sphere. R. B. Richter (private communication) conjectured the following:

Corollary 4.4. *Let x,y be elements of a locally connected, connected, compact subset M of the sphere.*

Then precisely one of the two statements (i),(ii) below holds:

(i): M has an SCC separating x and y.

(ii): The sphere contains an SA from x to y having only its ends in common with M.

Proof. We consider only the interesting case, namely when M is 3-connected. We consider M as an abstract metric space and add an SA denoted R between x and y. If the resulting space M_1 contains none of K_5 and $K_{3,3}$, then it can be embedded into the sphere, by Theorem 4.3. If we delete R, we obtain the embedding of M, by [4], as M is 3-connected. Hence (ii) holds.

On the other hand, if M_1 contains one of K_5 and $K_{3,3}$, then that subgraph contains R, and therefore it also contains an SCC satisfying (i). This completes the proof when M is 3-connected.

The case where M has smaller connectivity is tedious but straightforward. ∎

Adrien Douady (personal communication) has informed the author that Corollary 4.4 can also be obtained from first principles.

Finally, we discuss how to modify the obstructions in Theorem 4.3 when "2-connected" is replaced by "connected". We need the thumbtack as an obstruction. More generally, let D be any compact, locally connected, 3-connected subset of the 2-sphere. Then D has a unique embedding in the 2-sphere, by [4]. A point on a face boundary of D will be called a *boundary point of* D. Now define a *thumbtack with holes* as a space D' obtained by D by adding a line segment having precisely one end in common with D such that this end is not a boundary point of D. Clearly D' cannot be embedded in the 2-sphere.

It seems that an extension of the proof of Theorem 4.3 results in the following:

Theorem 4.5. *Let M be a connected, locally connected, compact topological space M. Then M is embeddable in the 2-sphere if and only if M is metrizable and contains none of the Kuratowski graphs K_5 and $K_{3,3}$ and no thumbtack with holes.*

To prove this result, some additional notation is convenient. We say that a subset B of M is a *block* of M if B is a maximal 2-connected subspace (with at least two elements). We say that an element x of M is a *cutpoint* if $M\backslash\{x\}$ is disconnected. One can prove that M has only countably many blocks $B_0, B_1, \ldots$ and that every block contains only countably many pointpoints of M. Moreover, two blocks have at most one element in common. Finally, we define a partial order on the blocks as follows: We say that B_0 is closer to B' than to B'' and write $B' < B''$, if every arc from B'' to B_0 intersects B'. We first embed B_0 in the plane. We add loops at the cutpoints in order to reserve space for the rest of M. Then we proceed to B_1 and those blocks that are between B_1 and B_0. Technical complications arise from the fact that there may be infinitely many such blocks.

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Carsten Thomassen

Department of Mathematics
Technical University of Denmark
DK-2800 Lyngby
Denmark
C.Thomassen@mat.dtu.dk